Let's Factor
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Divisibility, factoring, composing, decomposing – these are important themes in mathematics. This article discusses those topics, but the reason I am writing about methods to factor is simply for the fun of it. I mentally factor numbers into primes as a pastime while waiting in line or confined in car. (Yes, I am a math nerd.) As a bonus, I learn as I play. I'll start with basic techniques and then share the 150 Method.

Basic Techniques

The goal is to factor an integer. There is technology that will do this instantly for the 3- or 4-digit numbers I'm playing with, but where is the fun in that? Divisibility tests allow us to determine whether a number is a factor of our integer without doing the division.

Notation: We are testing X, an integer, for divisibility by n ($n \neq 0$ and is often a prime). The phrase "X is divisible by n" means specifically that the quotient of $\frac{X}{n}$ is an integer. Equivalent phrases include "n is a factor of X," "X is a multiple of n," and "n divides X." To apply the tests in Table 1, ask the given Question. The answer you determine is also the answer to, "Is X divisible by (Factor)?"

Table 1. Basic Divisibility Tests

Factor	Question	Examples
2	Is the last digit of X divisible by 2? That is, is the last digit 0, 2, 4, 6, or 8?	432: Yes!
5	Is the last digit of X divisible by 5? That is, is the last digit 0 or 5?	930 or 655: Yes and Yes!
3	Is the sum of the digits in X divisible by 3? (You may repeat summing the digits in the sum.)	87: $8 + 7 = 15$. Yes! Note that 3 divides $1 + 5$ also. 964: $9 + 6 + 4 = 14$. No. 6132897: $6+1+3+2+8+9+7 = 36$. Yes!
9	Is the sum of the digits in X divisible by 9? (You may repeat summing the digits in the sum.)	87: Sum is 15. No. 6132897: Sum is 36 (or $3 + 6 = 9$.) Yes!
4	Are the last two digits of X divisible by 4?	9088: Yes! 4 divides 88. 9930: No. 30 is not a multiple of 4.
11	Sum every other digit in X and subtract the sum of the remaining digits (or form an alternating sum: first digit$-$second digit$+$ third digit $-$ etc. It doesn't matter with which end you start.) Is the result divisible by 11?	154: $(1 + 4) - (5) = 0$ which is divisible by 11. Yes! 390907: $(3 + 0 + 0) - (9 + 9 + 7) = -22$. Yes! 11111: $1 - 1 + 1 - 1 + 1 = 1$. No.
8	Are the last 3 digits of X divisible by 8?	9088: Yes! 088 is divisible by 8. 9988: No. 988 is not divisible by 8.

The order of the factors in Table 1 are arranged (in my judgment) by ease to learn and apply. The test for divisibility by 8 is easy to remember, but harder to apply. It requires some number facts that are not (yet) automatic for me.

There are many more divisibility tests! See, for example, `https://en.wikipedia.org/wiki/Divisibility_rule`. You can learn variations of the tests as well as learn techniques for more divisors. (Proofs for the techniques are also cited.) I have not been willing to expend the brain power needed to remember other techniques. I am impressed by "mathemagicians" who astound audiences with tremendous feats of numerical agility, but I do not aspire to become one.

What do you notice?
Let's reflect. Pausing and looking for connections is what mathematical thinkers do! Here are a few things I noticed.

- The tests tell us about the remainder of a division problem (that is, whether or not it is 0) but do not give information about the quotient.
- Isn't it fun that the digits of a number divisible by 3 (or 9) can be rearranged willy-nilly and result in another number divisible by 3 (or 9)? What happens when the digits of a number divisible by 11 are rearranged? What rearrangements do you conjecture will work? Or what fraction of all the possible rearrangements will be divisible by 11?
- The tests given for powers of 2 extend. 2^k is a factor of X if 2^k is a factor of the last k digits of X. Why does this work? Do other tests extend?

The 150 Method

The goal is to factor a number into primes or ascertain that it is a prime itself. Recall we only need to check divisibility by primes up to the square root of the number. The tests in Table 1 (and a knowledge of number facts) are enough for two digit numbers. A basic strategy for larger numbers is to pull out the "easy" factors and manually check the quotient for divisibility by 7, 13, 17, 19, 23, and so forth. It works, but I'm reaching for my calculator and/or losing interest.

There is a nifty way to find more divisors. Mathematician and Field's medalist Dr. John Conway (1937-2020) shared the 150 Method on a MAA math history list-serve in 1995. I was intrigued and saved the post. This one method tests a number for nine primes, specifically, 2, 3, 5, 7, 11, 13, 17, 19, and 31. Learning this method seemed worth the effort! It does takes practice, but it can be done fairly easily, by hand. I'll describe the method and give examples, then discuss why it works.

Notation: The goal is to find prime factors of an integer X. To begin, we find how far away X is from a multiple of 150. We define A and K as follows:

$$X \pm A = 150K \qquad (A \text{ is positive}).$$

Use A and K to create a sequence of six numbers. Test each of the six numbers for divisibility by certain primes. Table 2 shows the sequence and which prime(s) the number is tested for.

If any of the six is divisible by its associated prime(s), then X is also divisible by those prime(s).

Table 2. 150 Method

Sequence	A	$A \pm 2K$	$A \pm 3K$	$A \pm 4K$	$A \pm 5K$	$A \pm 6K$
Prime(s)	2, 3, 5	19	17	7, 11	31	13

Examples

Example 1. $X = 511$.

511 is close to 450, which is $3 \cdot 150$. In fact, $511 - 61 = 3 \cdot 150$ which means $A = 61$ and $K = 3$. We will *subtract* multiples of 3 from 61 since 61 is subtracted from 511 to reach 450.

$A = 61$. Test 61 for divisibility by 2, 3, and 5.
$A - 2K = 55$. Test 55 for divisibility by 19.
$A - 3K = 52$. Test 52 for divisibility by 17.
$A - 4K = 49$. Test 49 for 7 and 11. Since 7 is a factor of 49, 7 is also a factor of 511.
$A - 5K = 46$. Test 46 for divisibility by 31.
$A - 6K = 43$. Test 43 for divisibility by 13.
One division gives the prime factorization: $511 = 7 \cdot 73$.

Example 2. $X = 731$.

731 is close to 750, which is $5 \cdot 150$. In fact, $731 + 19 = 5 \cdot 150$ so $A = 19$ and $K = 5$. We *add* multiples of 5 to 19 since 19 is added to 731 to reach a multiple of 150.

19 UP 5s	The sequence:	**19**	**29**	**34**	**39**	**44**	**49**
	Test primes:	(2, 3, 5)	19	17	(7, 11)	31	13

34 is divisible by 17 and hence, so is 731. The prime factorization is $731 = 17 \cdot 43$.

Dr. Conway recommended verbalizing the words/numbers **bolded** in the example. Further, touch your thumb when you say the second sequence number, your index finger when you say the third, and so on. You need to memorize the appropriate prime(s) for each step. I did this by writing them on my fingers when I practiced!

Example 3. $X = 949$.

900 is a convenient multiple of 150 close to 949. Here's the verbalization/action: **49 DOWN 6s**. (Since $949 - 49 = 6 \cdot 150$.) **49, 37**(touch thumb), **31** (touch index), **25** (touch middle), **19** (touch ring), **13 shows 13** (touch pinkie). 13 is a factor of 949 and $949/13 = 73$.

Example 4. $X = 851$.

49 UP 6s. 49, 61, 67, 73, 79, 85. No factors are shown. Since $\sqrt{851}$ is about 30, we need to manually check the two remaining primes smaller than 30, that is, 23 and 29. In fact, $851 = 23 \cdot 37$. There are only eight 3 digit numbers for which the 150 Method fails to find a factor. Dr. Conway shared a modification of the 150 method that is slightly more complicated but also tests for 23 and 29. With this modification, $37^2 = 1369$ is the first number for which the method fails to find a factor.

Example 5. $X = 693$. We will use basic techniques and then the 150 method.

Basic: 693 is divisible by 9 (9 divides $6+9+3$) and 11 (11 divides $6+3-9$).
150 method: 93 DOWN 4s: 93 (shows 3), 85, 81, 77 (shows 7 and 11), 73, 69.
In either case, after dividing 693 by the factors found, the prime factorization is $693 = 3 \cdot 3 \cdot 7 \cdot 11$.

Why does the 150 Method work?
In the 150 method, we want to know whether X is divisible by n, but we test a number added to, or subtracted from, X. So to understand the method, let's consider the possibilities when Z, a multiple of n, is decomposed into a sum or difference.

We start with two facts: $X \pm Y = Z$ and n divides Z.

Divide both sides of the equation by n: $\dfrac{X \pm Y}{n} = \dfrac{Z}{n}$. This is equivalent to

$$\frac{X}{n} \pm \frac{Y}{n} = \frac{Z}{n}.$$

Since n divides Z, then $\frac{Z}{n}$ is an integer. This also means that either (1) $\frac{X}{n}$ and $\frac{Y}{n}$ are both integers or that (2) neither are. It cannot happen that one is an integer and one is not! For example, 12 is divisible by 3, $9+3=12$ and both $9/3$ and $3/3$ are integers. Also, 12 is divisible by 4, and neither $9/4$ nor $3/4$ are integers.

In summary, we learned this FACT: When we know n is a divisor of Z, we can test one term of $X \pm Y$ for divisibility by n and know the answer applies to both terms.

Why do we use a sequence? The answer rests on the fact that all the primes we test for have a multiple close to 150. Scaling 150 by K scales these multiples as well.

Prime	Multiple
2, 3, 5	$150 = 2 \cdot 3 \cdot 5^2$
19	$152 = 19 \cdot 8$
17	$153 = 17 \cdot 9$
7, 11	$154 = 7 \cdot 11 \cdot 2$
31	$155 = 31 \cdot 5$
13	$156 = 13 \cdot 12$

The first step of the 150 Method was to decompose a multiple of 150 into a sum/difference using X (the number to factor). We wrote

$$X \pm A = 150K$$

Use the FACT from above: Since 2, 3, and 5 are factors of $150K$, if 2, 3, and/or 5 are factors of A, then they are also factors of X.

We also know 19 is a factor of $152K$. So add $2K$ to both sides of our starting equation:

$X + A = 150K$	or	$X - A = 150K$
$X + A + 2K = 152K$		$X - A + 2K = 152K$
$X + (A + 2K) =$ a multiple of 19		$X - (A - 2K) =$ a multiple of 19

Thus, we use the FACT to check $(A \pm 2K)$ for divisibility by 19 to learn whether or not X is divisible by 19. The rest of the method is explained in a similar fashion.

What do you notice? What do you wonder?

- Why does the sequence skip from A to $A \pm 2K$?
- What if you want to test for 8, 9, or another number with the 150 Method? Can you?
- What challenges do you encounter performing the 150 Method?
- Is there a biggest number for which the method works? If not, is there a point where the method is not practical?

What do you think? Are you tempted to learn the 150 Method? Dr. Conway closed his post by saying, "I hope someone out there finds this useful. It really DOES make factorizing very easy." Yes, Dr. Conway, I enjoy using your method! I have developed my number sense, exercised my memory, and deepened my understanding of mathematics. I'm sharing it now with the same hope – that another reader out there will find it useful, even if just for fun.